

# A Planar Lattice Graph, with Empty Intersection of All Longest Path

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**Abstract:** Tibor Gallai in 1966 elevated the declaration about the existence of graphs with the property that every vertex is missed by some longest path. This property will be called Gallai's property. First answer back by H. Walther, who introduced a planar graph on 25 vertices satisfying Gallai's property, and various authors worked on that property, after examples of such graphs were found while examining such  $n$ -dimensional  $L^n$  graphs with the property that every longest Paths have empty intersection, can be embeddable in  $\mathbb{R}^n$ , Some in equilateral triangular lattice  $T$ , Square lattice  $L^2$ , hexagonal lattice  $H$ , also on the torus, Mobius strip, and the Klein bottle but no hypo-Hamiltonian graphs are embeddable in the planar square lattice. In this paper we present a graph embeddable into Cubic lattices  $L^3$ , such that graphs can also occur as sub graphs of the cubic lattices, and enjoying the property that every vertex is missed by some longest path. Here research has also significance in applications. What if several processing units are interlinked as parts of a lattice network. Some of them developing a chain of maximal length are used to solve a certain task. To get a self-stable fault-tolerant system, it is indispensable that in case of failure of any unit or link, another chain of same length, not containing the faulty unit or link, can exchange the chain originally used. This is exactly the case investigated here. We denote by  $L^n$  the  $n$ -dimensional cubic lattice in  $\mathbb{R}^n$ .

**Keywords:** Longest Paths, Longest Cycles, Lattice Graphs, Cubic Lattices, Gallai's Property

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## 1. Introduction

A graph  $G$  is *Hamiltonian* if there exists a *Hamiltonian cycle* in  $G$ , i.e. a cycle which passes through every vertex of graph  $G$ . A *Hamiltonian path* of a graph  $G$  is a path in  $G$  that contains all vertices of graph  $G$ . A graph  $G$  with a Hamiltonian path is called *traceable*. A non-Hamiltonian graph  $G$  such that  $G - v$  is Hamiltonian for every vertex  $v$  is said to be *hypo-Hamiltonian*. Petersen's graph is a well-known example of a hypo-Hamiltonian graph.

The presence of hypo-Hamiltonian graphs and earlier the modernization of the hypo traceable graphs, in 1966 T. Gallai [1] (Tibor Gallai was a Hungarian mathematician. He worked in combinatorics, especially in graph theory, and was a lifelong friend and collaborator of Paul Erdős. He was a student of Dénes König and an advisor of László Lovász. He was a corresponding member of the Hungarian Academy of Sciences) asked whether there exist connected graphs with the property that every vertex is missed by some longest path. Just later, in 1969, Gallai's question was first responded by H.

Walther [2], who introduced a planar graph  $G$  on 25 vertices satisfying Gallai's property. Walther's example had connectivity 1. In this context appeared T. Zamfirescu's questions about the existence of (small, if possible minimal)  $i$ -connected graphs with the property that for any  $j$  vertices there is a longest path avoiding all of them, it was asked about examples with higher connectivity [3]. Such graphs have been subsequently found, up to connectivity number 3 [4, 5, 6]. The problem whether 4-connected graphs with the property asked by Gallai (with respect to paths or cycles) do or do not exist is still unsolved. In this paper, given graph is connected.

There exist large classes of graphs without Gallai's property. S. Klavžar and M. Petkovšek proved that split graphs and cacti are among them [7].

B. Menke [8] considered the following class of graphs. In the usual infinite planar square lattice  $L^2$  consider a finite cycle and all vertices and edges lying on or inside that cycle. Such graphs are called grid or mesh graphs, and B. Menke proved that no grid graph has Gallai's property. See also [5]. Our aim here is to be responsible for examples of graphs with

Gallai's property embeddable in higher-dimensional ( $L^3$ ) lattices.

In 2012, AD Jumani and T. Zamfirescu [9] set up graphs which are embeddable into the equilateral triangular lattice, recently, A. Shabbir [10] introduced a graph  $G$  on 58 vertices embedded on in a hexagonal lattice on the Klein bottle, a planar graph  $G$  on 17 vertices embedded in a square lattice on the Klein bottle, a planar two-connected graph  $G$  on 80 vertices embedded in a square lattice on the Klein bottle, a graph  $G$  on 12 vertices in a triangular lattice on the Klein bottle, and a planar two-connected graph  $G$  on 48 vertices in a triangular lattice on the Klein bottle. Y. Bashir, F. Nadeem, and A. Shabbir [11] introduced connected graphs such that every pair of vertices is missed by a longest path in the triangular, square, and hexagonal lattices. They also found such graphs in some lattices embedded on the torus, Mobius strip, and the Klein bottle.

## 2. Embeddings of Graph with Empty Intersection of All Longest Paths

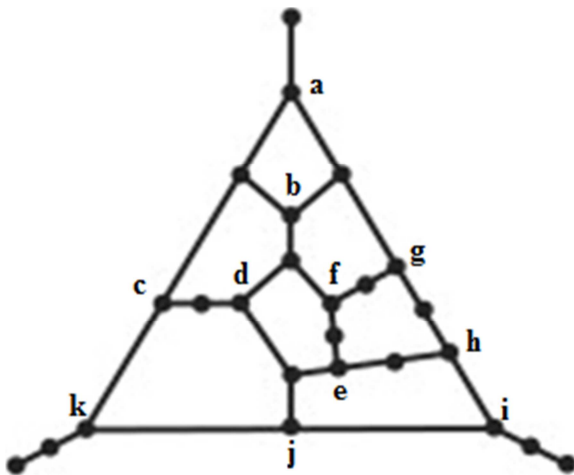


Figure 1. H. Walther's Planar Graph on 25 vertices

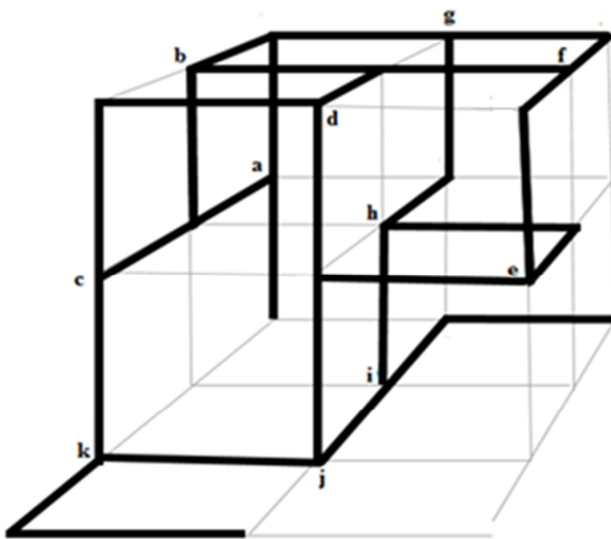


Figure 2. Embedding in cubic lattices.

In the part we dealt with graphs satisfying Gallai's property as regards longest paths. The smallest known example of a graph, the longest paths of which have empty intersection, without any restrictions, was independently exhibited by Voss and Walther [12], and Tudor Zamfirescu [13]. Frustrating to use it, we found the graph of Figure 2, homeomorphic to the graph of figure 1. It also enjoys the stated property and, moreover, is embeddable in the 3-dimensional cubic lattice. It has order 25. However, another graph, Schmitz' graph [14], the smallest known planar graph with Gallai's property with respect to paths, is better suitable, having smaller order 17. While Schmitz' graph [14] is not embeddable in  $L^2$ , it is embeddable in  $L^3$  [15]. While H. Walther's graph [2] is not embeddable in  $L^2$ , it acknowledges an embedding in  $L^3$ !

**Theorem.** There exists a 1-connected graph of order 25, imbedded in  $L^3$  whose longest paths have empty intersection.

**Proof:** Consider H. Walther's [2] graph  $G$  in Figure 1. The order of  $G$  is 25, and for every vertex there exists some longest path of length 21 avoiding it, and let  $G'$  be a subgraph in figure 2. Graph  $G$  is homeomorphic to the graph  $G'$  for each edge of  $G$ , the corresponding path of  $G$ , as well as numbers of vertices of degrees, shown in figure 2, appearances embedding of  $G$  in  $L^3$ .

## 3. Conclusion

We have described cubic lattice graph our implementation as a drawing of the corresponding graph, embedded into a cubic lattice, graph  $G$  is homeomorphic to the graph  $G'$  for each edge of  $G$ , the corresponding path of  $G$ , as well as number of vertices of degree shown in figure 2, appearances cubic lattice of  $G$ .

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