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$Z_2(Z_2+uZ_2)(Z_2+uZ_2+u^2Z_2)$-Additive Cyclic Codes

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Abstract: In this paper, we introduce the algebraic structure of Z_2 (Z_2+uZ_2) $(Z_2+uZ_2+u^2Z_2)$ -additive codes and Z_2 (Z_2+uZ_2) $(Z_2+uZ_2+u^2Z_2)$ -additive cyclic codes. Compared to the $Z_2Z_4Z_8$-additive codes, the Gray image of any Z_2 (Z_2+uZ_2) $(Z_2+uZ_2+u^2Z_2)$ -linear code will always be a linear binary code. Therefore, we consider the Z_2 (Z_2+uZ_2) $(Z_2+uZ_2+u^2Z_2)$ -additive cyclic codes as a $(Z_2+uZ_2+u^2Z_2)$ $[x]$ -submodule of $Z_2^{\alpha}\times(Z_2+uZ_2)^{\beta}\times(Z_2+uZ_2+u^2Z_2)^{\theta}$. We give the definition of Z_2 (Z_2+uZ_2) $(Z_2+uZ_2+u^2Z_2)$ -additive codes with generator matrices and parity-check matrices. Furthermore, we give the fundamental result on considering their additive cyclic codes with generator polynomials and spanning sets.

Keywords: Additive Codes, Cyclic Codes, Minimal Generating Set

1. Introduction

Codes over rings were introduced in early 1970s. However this research topic had been widely and really concerned after a milestone paper written by Hammons et al. in 1994, which shown that some special classes of non-linear binary codes could be obtained as Gray images of linear codes over Z_4 [1]. Since then, several families of codes over finite rings have been studied by many scientists [2-4].

In 1973, Delsarte defined additive codes in terms of association schemes as the subgroups of the underlying abelian group [5]. In 2010, Borges et al. brought a new perspective to additive codes over rings, introducing Z2Z4-additive codes [6]. Under binary Hamming scheme, the underlying group of order 2^k isomorphic to $Z_2^{\alpha}\times Z_4^{\beta}$, where α and β are nonnegative integers. The subgroups of underlying group are called Z_2Z_4-additive codes. Therefore, a Z_2Z_4-additive code C is defined to be a subgroup of $Z_2^{\alpha}\times Z_4^{\beta}$, where $\alpha+2\beta=n$. If $\alpha=0$, then C are quaternary linear codes over Z_4 and if $\beta=0$, C are equivalent to binary linear codes. In 2014, Abualrub et al. studied Z_2Z_4 -additive cyclic codes and Z_2+uZ_2 -linear cyclic codes [7]. In 2015, Aydogdu et al. studied the $Z_2Z_2[u]$ -additive codes [8]. In 2016, Borges et al. researched the generator polynomials and dual codes of Z_2Z_4 -additive codes [9]. One year later, Aydogdu et al. studied the $Z_2Z_4Z_8$ -cyclic codes [10], in which they introduced that if α, β and θ are odd integers, $Z_2Z_4Z_8$-cyclic codes is a Z_8 -submodule of $Z_2/(x^{\alpha}-1)\times Z_4/(x^{\beta}-1)\times Z_8/(x^{\theta}-1)$ and give the minimal generating set for $Z_2Z_4Z_8$-cyclic code. In 2018, the binary images of Z_2Z_4-additive cyclic codes have been done also by Borges et al. [11]. Further, Ismail et al. discussed the structure properties of Z2Z2s-additive cyclic codes [12].

In this paper, we will study some structure of $Z_2(Z_2+uZ_2)(Z_2+uZ_2+u^2Z_2)$ -additive cyclic codes, including their generator polynomials and spanning sets. The rest of this paper is organized as follows. In Section 2, some results on $Z_2(Z_2+uZ_2)(Z_2+uZ_2+u^2Z_2)$ -additive code are given, and associate these codes with submodules of $Z_2^{\alpha}\times(Z_2+uZ_2)^{\beta}\times(Z_2+uZ_2+u^2Z_2)^{\theta}$. Moreover, the generator matrices and parity-check matrices of $Z_2(Z_2+uZ_2)(Z_2+uZ_2+u^2Z_2)$ -additive codes are given in this section. In Section 3, the definition $Z_2(Z_2+uZ_2)(Z_2+uZ_2+u^2Z_2)$ -additive cyclic codes and their generator polynomials and spanning sets are given. Some examples are given to illustrate the main results appeared in this paper. Moreover, using the Gray map, some good binary linear codes can be obtained by

$Z_2(Z_2+uZ_2)(Z_2+uZ_2+u^2Z_2)$-additive cyclic codes.

2. $Z_2(Z_2+uZ_2)(Z_2+uZ_2+u^2Z_2)$-Additive Codes

Let Z_2 be the finite field of order 2, and let R be the ring $Z_2+uZ_2=\{0,1,u,1+u\}$, where $u^2=0$ mod 2, and let S be the ring $Z_2+uZ_2+u^2Z_2=\{0,1,u,1+u,u^2,1+u^2,u+u^2,1+u+u^2\}$, where $u^2=0$ mod 2. Construct the following set

$$\begin{aligned}\Re &= Z_2^{\alpha}\times(Z_2+uZ_2)^{\beta}\times(Z_2+uZ_2+u^2Z_2)^{\theta}\\ &=\left\{(m,v,w)\mid m\in Z_2^{\alpha}, v\in(Z_2+uZ_2)^{\beta}, w\in(Z_2+uZ_2+u^2Z_2)^{\theta}\right\}\end{aligned} \quad (1)$$

Clearly, this set is closed under usual addition and it becomes an additive abelian group. Define the following multiplication for $(m,v,w)\in\Re$ and $d\in S$, in order to make $\Re$ closed under multiplication by elements from S,

$$d\cdot(m,v,w)=(dm \bmod u, dv \bmod u^2, dw) \quad (2)$$

Since $d\cdot(m,v,w)\in S$, it follows that $\Re$ is a S-module with respect to this scalar multiplication.

Definition 1 A non-empty subset C of $Z_2^{\alpha}\times R_2^{\beta}\times S_2^{\theta}$ is called a Z_2RS-additive codes if it is a subgroup of $Z_2^{\alpha}\times R_2^{\beta}\times S_2^{\theta}$.

Clearly, C is a binary linear code of length α if $\beta=0$ and $\theta=0$, a Z_2R-additive code of length (α,β) if $\theta=0$ and a linear code of length θ over S if $\alpha=0$ and $\beta=0$. The type of a Z_2RS-additive code of block length (α,β,θ) is defined as follows.

Definition 2 A Z_2RS-additive code C of length (α,β,θ) is called Z_2RS-additive code of type ($\alpha,\beta,\theta;k_0;k_1,k_2;k_3,k_4,k_5$), if C is a group isomorphic to the abelian structure

$$Z_2^{k_0}\times R^{k_1}\times Z_2^{l_2}\times S^{k_3}\times R^{k_4}\times Z_2^{k_5} \quad (3)$$

From [13, 14], defind $\phi_1: Z_2+uZ_2\to Z_2^2$ by $\phi_1(0)=(0,0)$, $\phi_1(1)=(0,1)$, $\phi_1(u)=(1,1)$, $\phi_1(1+u)=(1,0)$ and $\phi_2: Z_2+uZ_2+u^2Z_2\to Z_2^3$ by $\phi_2(0)=(0,0,0)$, $\phi_2(1)=(0,1,1)$, $\phi_2(u)=(0,0,1)$, $\phi_2(1+u)=(0,1,0)$, $\phi_2(u^2)=(1,1,0)$, $\phi_2(1+u^2)=(1,0,1)$, $\phi_2(u+u^2)=(1,1,1)$, $\phi_2(1+u+u^{2)}=(1,0,0)$.

By the maps ϕ_1 and ϕ_2, Gray map can be defined by $\Phi: Z_2^{\alpha}\times R^{\beta}\times S^{\theta}\to Z_2^n$ by:

$$\Phi(m,v,w)=(m_0,\dots,m_{\alpha-1},\phi_1(v_0),\dots,\phi_1(v_{\beta-1}),\phi_2(w_0),\dots,\phi_2(w_{\theta-1})) \quad (4)$$

where

$$m=(m_0,m_1,\dots,m_{\alpha-1})\in Z_2^{\alpha},\ v=(v_0,v_1,\dots,v_{\beta-1})\in(Z_2+uZ_2)^{\beta} \quad (5)$$

$$w=(w_0,w_1,\dots,w_{\theta-1})\in(Z_2+uZ_2+u^2Z_2)^{\theta} \quad (6)$$

Definition 3 The Gray image $\Phi(C)=C$ of a Z_2RS-additive code C is a binary code of length $n=\alpha+2\beta+3\theta$ and is called a Z_2RS-linear code.

Let F_q is the finite field with q elements, a linear code of length n over F_q is a linear subspace of the vector space $F_q{}^n$.

The generator matrix of linear codes can be formed by the minimal spanning set of this linear code. Since the minimum spanning set of the linear codes is not unique, then the generator matrix of the linear codes is not unique. In this following, the paper gives the standard form of generator matrix of Z_2RS-additive codes first.

Theorem 1 Let C be a Z_2RS-additive code of type ($\alpha,\beta,\theta;k_0;k_1,k_2;k_3,k_4,k_5$). Then C is permutation equivalent to a Z_2RS-additive code with the standard form matrix.

$$G_s=\left(\begin{array}{cc|ccc|cccc} I_{k_0} & A'_{01} & 0 & 0 & uT_1 & 0 & 0 & 0 & u^2T_2\\ 0 & S'_1 & I_{k_0} & B_{01} & B_{02} & 0 & 0 & uT_3 & uT_4\\ 0 & 0 & 0 & uI_{k_2} & uB_{12} & 0 & 0 & 0 & u^2T_5\\ 0 & S'_2 & 0 & S_{01} & S_{02} & I_{k_3} & A_{01} & A_{02} & A_{03}\\ 0 & S'_3 & 0 & 0 & uS_{12} & 0 & uI_{k_4} & uA_{12} & uA_{13}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u^2I_{k_5} & u^2A_{23}\end{array}\right) \quad (7)$$

where A'_{01}, S'_1, S'_2 and S'_3 are Z_2 matrices. B_{02}, B_{12}, S_{02}, S_{12} and B_{01}, S_{01}, T_1 are R-matrices, T_4, T_5, A_{03}, A_{13}, A_{23} and T_2, T_3, A_{01}, A_{02}, A_{12} are S-matrices. Further, the number of codewords in C is given by:

$$|C|=2^{k_0}\cdot4^{k_1}\cdot2^{k_2}\cdot8^{k_3}\cdot4^{k_4}\cdot2^{k_5} \quad (8)$$

Define a inner product for the elements $u,v\in Z_2^{\alpha}\times R^{\beta}\times S^{\theta}$ as

$$u\cdot v=u^2\left(\sum_{i=1}^{\alpha}u_iv_i\right)+u\left(\sum_{j=\alpha+1}^{\alpha+\beta}u_jv_j\right)+\sum_{k=\alpha+\beta+1}^{\alpha+\beta+\theta}u_kv_k \quad (9)$$

The dual code $C^{\perp}$ can be defined in usual way with respect to the inner product.

$$C^{\perp}=\left\{u\in Z_2^{\alpha}\times R^{\beta}\times S^{\theta}\mid u\cdot v=0\, for all u\in C\right\} \quad (10)$$

It is obviously that $C^{\perp}$ is also a Z_2RS -additive code.

Corollary 1 Let C be a Z_2RS -additive code of type with standard form generator matrix (7). Then, the code $C^{\perp}$ is of type

$$(\alpha,\beta,\theta;\alpha-k_0;\beta-k_1-k_2,k_2;\theta-k_3-k_4-k_5,k_5,k_4) \quad (11)$$

3. Z2RS-Additive Cyclic Codes

Cyclic codes are a very significant class of linear codes because of their good algebraic structures for coding and decoding [15]. In this section, we study the structural properties of Z_2RS -additive cyclic codes, including their generator polynomials and minimal spanning sets.

Let δ be a standard shift operator on Z_2^{α}, R^{β} and S^{θ}. For any $(m,v,w)\in Z_2^{\alpha}\times R^{\beta}\times S^{\theta}$, let τ be a shift operator on

$$Z_2^{\alpha}\times R^{\beta}\times S^{\theta} \text{ as } \tau(m,v,w)=(\delta(m),\delta(v),\delta(w)) \quad (12)$$

Definition 4 The Z_2RS -additive code C of length $n=\alpha+\beta+\theta$ over $\Re$ is said to be a Z_2RS -additive cyclic code if it is invariant under τ, i.e. for any codeword,

$$c=\left(m_0,m_1,\cdots,m_{\alpha-1},v_0,v_1,\cdots,v_{\beta-1},w_0,w_1,\cdots,w_{\theta-1}\right)\in C \quad (13)$$

its cyclic shift

$$\tau(c)=\left(m_{\alpha-1},m_0,\cdots,m_{\alpha-2},v_{\beta-1},v_0,\cdots,v_{\beta-2},w_{\theta-1},w_0,\cdots w_{\theta-2}\right)\in C \quad (14)$$

In Section 2, the paper defined the inner product in $Z_2^{\alpha}\times R^{\beta}\times S^{\theta}$.

Lemma 1 If C is a Z_2RS -additive cyclic code, then $C^{\perp}$ is also a Z_2RS -additive cyclic code.

Proof Let C be a Z_2RS -additive cyclic code, and

$$m=\left(a_0,\cdots a_{\alpha-}1,b_0,\cdots,b_{\beta-1},d_0,\cdots d_{\theta-1}\right)\in C^{\perp} \quad (15)$$

In the following, it only needs to prove $\tau(m)\in C^{\perp}$. Since $m\in C^{\perp}$, for

$$v=(e_0,\cdots,e_{\alpha-1},g_0,\cdots,g_{\beta-1},h_0,h_{\theta-1})\in C \quad (16)$$

it have

$$\begin{aligned}&m\cdot v=u^2(a_0e_0+\cdots+a_{\alpha-1}e_{\alpha-1})+u(b_0g_0+\cdots+b_{\beta-1}g_{\beta-1})\\&+d_0h_0+\cdots+d_{\theta-1}h_{\theta-1}\equiv 0\bmod u^3.\end{aligned} \quad (17)$$

Let

$$l=lcm(\alpha,\beta,\theta) \quad (18)$$

and

$$\tau^{l-1}(v)=(e_1,\ldots,e_{\alpha-1},e_0,g_1,\ldots,g_{\beta-1},g_0,h_1,\ldots,h_{\theta-1},h_0)=w \quad (19)$$

Then $\tau^l=v$. Since C is a cyclic code, it follows that $w\in C$. Therefore,

$$\begin{aligned}&0=w\cdot m\\&=u^2(e_1a_0+\cdots+e_{\alpha-1}a_{\alpha-2}+e_0a_{\alpha-1})+u(g_1b_0+\cdots+g_0b_{\beta-1})\\&+h_1d_0+h_0d_{\theta-1}\\&=u^2(e_0a_{\alpha-1}+\cdots+e_{\alpha-1}a_{\alpha-2})+u(g_0b\beta-1++g_{\beta-1}b_{\beta-2})\\&+h_0d_{\theta-1}++h_{\theta-1}d_{\theta-2}\\&=v\cdot\tau(m).\end{aligned} \quad (20)$$

Hence, $\tau(u)\in C$, $C^{\perp}$ is also a Z_2RS -additive cyclic code.

Definition 5 Define the set $Z_2[x]/(x^{\alpha}-1)\times R[x]/(x^{\beta}-1)\times S[x]/(x^{\theta}-1)$ by $\Re_{\alpha,\beta,\theta}$. For $C\subseteq Z_2^{\alpha}\times R^{\beta}\times S^{\theta}$ any elenment:

$$c=\left(m_0,m_1,\cdots,m_{\alpha-1},v_0,v_1,\cdots,v_{\beta-1},w_0,w_1,\cdots,w_{\theta-1}\right)\in C,$$

can be written by the element of $\Re_{\alpha,\beta,\theta}$ as follows

$$\begin{aligned}&c(x)=(m_0+m_1x+\cdots+m_{\alpha-1}x^{\alpha-1},v_0+v_1x+\cdots+v_{\beta-1}x^{\beta-1},\\&w_0+w_1+\cdots+w_{\theta-1}x^{\theta-1})=(m(x),v(x),w(x)).\end{aligned} \quad (21)$$

Therefore, it is possible to see that $\Re_{\alpha,\beta,\theta}\cong Z_2^{\alpha}\times R^{\beta}\times S^{\theta}$. Define the product $*$, for $d(x)\in S[x]$ and $(f(x),g(x),h(x))\in\Re_{\alpha,\beta,\theta}$,

$$\begin{aligned}&d(x)*(f(x,)g(x),h(x))\\&=(d(x)f(x)\bmod u,d(x)g(x)\bmod u^2,d(x)h(x))\end{aligned} \quad (22)$$

This extended multiplication is also well defined and $Z_2^{\alpha}\times R^{\beta}\times S^{\theta}$ is a $S[x]$ -module with respect to this multiplication.

Lemma 2 Under the definition of $*$, $\Re_{\alpha,\beta,\theta}$ is an $S[x]$-module, and any Z_2RS -additive cyclic code C corresponds to an $S[x]$-submodule of $\Re_{\alpha,\beta,\theta}$.

In this part, we study $S[x]$-submodules for $\Re_{\alpha,\beta,\theta}$. The generator and a minimal spanning set of these submodules are determined. Assume that α, β, and θ are odd positive integers. Since the Z_2RS -additive cyclic code C is an $S[x]$

-submodule of $\Re_{\alpha,\beta,\theta}$, a map can be defined as follows

$$\begin{aligned}&\psi: C \to S[x]/(x^{\theta}-1)\\&(f(x),g(x),h(x)) \to h(x)\end{aligned} \quad (23)$$

Clearly, ψ is an $S[x]$-module homomorphism with kernel

$$\begin{aligned}&Ker(\psi)=\{(f(x),g(x),0)\in C: f(x)\in Z_2[x]/(x^{\alpha}-1),\\&g(x)\in R[x]/(x^{\beta}-1)\}\end{aligned} \quad (24)$$

Since the image $\psi(C)$ of C is an ideal of $S[x]/(x^{\theta}-1)$. and θ is an odd integer, which implies that

$$\psi(C)=\left\langle p(x)+uq(x)+u^2r(x)\right\rangle \quad (25)$$

where $r(x)\,|\,q(x)\,|\,p(x)\,|\,(x^{\theta}-1) \bmod u^3$.

From the discussion above, which implies that the following theorem is got.

Theorem 3 Let C be a Z_2RS-additive cyclic code. Then C can be generated as an $R[x]$-submodule of $\Re_{\alpha,\beta,\theta}$ with this form

$$\begin{aligned}C=\langle\ &(f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle.\end{aligned} \quad (26)$$

Proof Let I be a set, and

$$\begin{aligned}&I=\{(f(x),g(x),0)\in Z_2[x]/(x^{\alpha}-1)\times R[x]/(x^{\beta}-1):\\&(f(x),g(x),0)\in \ker(\psi)\}\end{aligned} \quad (27)$$

Clearly, I is an $R[x]$-submodule of $Z_2[x]/\left\langle x^{\alpha}-1\right\rangle\times R[x]/\left\langle x^{\beta}-1\right\rangle$. Hence, I is a Z_2RS-additive cyclic code. From [7],

$$I=\langle\ (f(x),0),(f_1(x),g_1(x)+ua_1(x)\ \rangle, \quad (28)$$

where $f(x)\,|\left\langle x^{\alpha}-1\right\rangle$ and $g_1(x)+ua_1(x)$ is a polynomial over R with $a_1(x)\,|\,g_1(x)\,|\left\langle x^{\beta}-1\right\rangle$ and $f(x)\,|\,\frac{x^{\beta}-1}{a_1(x)}f_1(x)$.

Let $(c_1(x),c_2(x),0)\in \ker(\psi)$. Then that have

$$\begin{aligned}&(c_1(x),c_2(x))\in I\\&=\left\langle (f(x),0),(f_1(x),g_1(x)+ua_1(x))\right\rangle\end{aligned} \quad (29)$$

Therefore, for polynomials $m_1\in Z_2[x]/\left\langle x^{\alpha}-1\right\rangle$ and $m_2\in R[x]/\left\langle x^{\beta}-1\right\rangle$, it get

$$(c_1(x),c_2(x))=m_1*(f(x),0)+m_2*(f_1(x),g_1(x)+ua_1(x)) \quad (30)$$

So

$$\ker(\psi)=\left\langle (f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0\right\rangle \quad (31)$$

is a submodule of C. By the first isomorphism theorem of modules, there are $C/\ker(\psi)\cong\left\langle p(x)+uq(x)+u^2r(x)\right\rangle$.

Let $(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\in C$ such that

$$\begin{aligned}&\psi(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\\&=p(x)+uq(x)+u^2r(x)\end{aligned} \quad (32)$$

Then

$$\begin{aligned}C=\langle\ &(f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle.\end{aligned} \quad (33)$$

Lemma 3 Let

$$\begin{aligned}C=\langle\ &(f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle\end{aligned} \quad (34)$$

be a Z_2RS-additive cyclic code. Then

$$\deg f_1(x)<\deg f(x) \quad (35)$$

and $\deg f_2(x)<\deg f(x)$ and

$$\deg g_2(x)<\deg g_1(x) \quad (36)$$

Proof Let $\deg f_1(x)\ge \deg f(x)$. Since $f(x)$ is monic, it can apply division algorithm, i.e. there exist polynomials $q'(x)$ and $r'(x)$ over Z_2 such that

$$f_1(x)=f(x)q'(x)+r'(x) \quad (37)$$

where $r'(x)=0$ or $0\le \deg r'(x)<\deg f(x)$, which implies that

$$\begin{aligned}&\langle\ (f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle\\&=\langle\ (f(x),0,0),(f(x)q'(x)+r'(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle\\&=\langle\ (f(x),0,0),(r'(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle.\end{aligned} \quad (38)$$

Hence we may assume that $\deg f_1(x)<\deg f(x)$. Similarly the $\deg f_2(x)<\deg f(x)$ and $\deg g_2(x)<\deg g_1(x)$ also can be proved.

Lemma 4 Let

$$\begin{aligned}C=\langle\ &(f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle.\end{aligned} \quad (39)$$

be a Z_2RS-additive cyclic code. Then

$$f(x)\mid\frac{x^{\beta}-1}{a_1(x)}f_1(x) \quad (40)$$

Proof It is well know that

$$\psi(C)=\langle p(x)+uq(x)+u^2r(x)\rangle \quad (41)$$

Hence,

$$\begin{aligned}&\psi\left(\frac{x^{\beta}-1}{a_1(x)}*(f_1(x),g_1(x)+ua_1(x),0)\right)\\&=\psi\left(\delta\left(\frac{x^{\beta}-1}{a_1(x)}\right)f_1(x),\delta\left(\frac{x^{\beta}-1}{a_1(x)}\right)(g_1(x)+ua_1(x)),0)\right)\\&=0\end{aligned} \quad (42)$$

where $(f_1(x),g_1(x)+ua_1(x),0)\in C$, which implies that

$$\delta\left(\frac{x^{\beta}-1}{a_1(x)}\right)f_1(x)=\left(\frac{x^{\beta}-1}{a_1(x)}\right)f_1(x)\in\ker(\psi) \quad (43)$$

Therefore,

$$f(x)\mid\frac{x^{\beta}-1}{a_1(x)}f_1(x) \quad (44)$$

Lemma 5 Let

$$\begin{aligned}C=\langle\ &(f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle.\end{aligned} \quad (45)$$

be a Z_2RS -additive cyclic code. Then have

$$(g_1(x)+ua_1(x))\mid\frac{x^{\theta}-1}{r(x)}g_2(x)\bmod u^2 \quad (46)$$

$$f(x)\mid(k(x)f_1(x)+\frac{x^{\theta}-1}{r(x)}f_2(x))\bmod u \quad (47)$$

Where

$$k(x)(g_1(x)+ua_1(x))=\frac{x^{\theta}-1}{r(x)}g_2(x)\bmod u^2 \quad (48)$$

Proof (i) Consider

$$\begin{aligned}&\psi\left(\frac{x^{\theta}-1}{r(x)}*(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\right)\\&=\psi\left(\delta\left(\frac{x^{\theta}-1}{r(x)}\right)f_2(x),\delta\left(\frac{x^{\theta}-1}{r(x)}\right)g_2(x),0)\right)\\&=0.\end{aligned} \quad (49)$$

Since

$$\delta\left(\frac{x^{\theta}-1}{r(x)}\right)g_2(x)=\frac{x^{\theta}-1}{r(x)}g_2(x)\in\ker(\psi) \quad (50)$$

then

$$(g_1(x)+ua_1(x))\mid\frac{x^{\theta}-1}{r(x)}g_2(x)\bmod u^2 \quad (51)$$

(ii) Let

$$(g_1(x)+ua_1(x))\mid\frac{x^{\theta}-1}{r(x)}g_2(x)\bmod u^2 \quad (52)$$

Then there exists $k(x)\in S$ such that

$$k(x)(g_1(x)+ua_1(x))=\frac{x^{\theta}-1}{r(x)}g_2(x)\bmod u^2 \quad (53)$$

Having

$$\begin{aligned}&k(x)*(f_1(x),g_1(x)+ua_1(x),0)\\&=(k(x)f_1(x),k(x)(g_1(x)+ua_1(x),0))\in C\end{aligned} \quad (54)$$

Therefore,

$$\begin{aligned}&(k(x)f_1(x)+k(x)(g_1(x)+ua_1(x)),0)\\&-(\frac{x^{\theta}-1}{r(x)}f_2(x),\frac{x^{\theta}-1}{r(x)}g_2(x),0)\\&=(k(x)f_1(x)+\frac{x^{\theta}-1}{r(x)}f_2(x),0,0)\in\ker(\psi)\subseteq C.\end{aligned} \quad (55)$$

Hence,

$$f(x)\mid(k(x)f_1(x)+\frac{x^{\theta}-1}{r(x)}f_2(x))\bmod u \quad (56)$$

From the discussion above, the following theorem can be directly obtained.

Theorem 4 Let C be a Z_2RS -additive cyclic code. Then,

$$C=\langle(f(x),0,0)\rangle, f(x)\mid(x^{\alpha}-1)\bmod u \quad (57)$$

or

$$C=\langle f_1(x),g_1(x)+ua_1(x),0\rangle\ ,\ a_1(x)\mid g_1(x)\mid(x^{\beta}-1)\bmod u^2$$

and

$$(x^{\alpha}-1)\mid(\frac{x^{\beta}-1}{a_1(x)})f_1(x) \quad (58)$$

or

$$\begin{aligned}C=\langle\ &(f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\\&(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle,\end{aligned} \quad (59)$$

where

$$r(x)\mid q(x)\mid p(x)\mid (x^{\theta}-1)\bmod u^3 \quad (60)$$

$$\deg g_2(x)<\deg g_1(x) \quad (61)$$

and

$$(g_1(x)+ua_1(x))\mid \frac{x^{\theta}-1}{a_1(x)}g_2(x)\bmod u^2 \quad (62)$$

$$\deg f_1(x)<\deg f(x), \deg f_2(x)<\deg f(x) \quad (63)$$

and

$$f(x)\mid (\frac{x^{\beta}-1}{a_1(x)})f_1(x)\bmod u \quad (64)$$

$$f(x)\mid (k(x)f_1(x)+\frac{x-1}{r(x)}f_2(x))\bmod u \quad (65)$$

where

$$k(x)(g_1(x)+a_1(x))=\frac{x^{\theta}-1}{r(x)}g_2(x)\bmod u^2 \quad (66)$$

In the following, get the main result on the minimum generating sets of Z_2RS -additive cyclic codes.

Theorem 5 Let

$$\begin{aligned}C=\langle\ &(f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\\ &(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle.\end{aligned} \quad (67)$$

Let,

$$\begin{aligned}
D_1&=\bigcup_{i=0}^{\alpha-t_1-1}\left\{x^i*\left(f(x),0,0\right)\right\},\\
D_2&=\bigcup_{i=0}^{\beta-t_2-1}\left\{x^i*\left(f_1(x),g_1(x)+ua_1(x),0\right)\right\},\\
D_3&=\bigcup_{i=0}^{\theta-t_3-1}\left\{x^i*\left(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x)\right)\right\},\\
D_4&=\bigcup_{i=0}^{\deg(a_1'(x))-1}\left\{x^i*\left(l_{g1}(x)f_1(x),ul_{g1}(x)a_1(x),0\right)\right\},\\
D_5&=\bigcup_{i=0}^{\deg(q'(x))-1}\left\{x^i*\left(l_p(x)f_2(x),l_p(x)g_2(x),l_p\left(uq(x)+u^2r(x)\right)\right)\right\},\\
D_6&=\bigcup_{i=0}^{\deg(r'(x))-1}\left\{x^i*\left(l_p(x)f_2(x),l_p(x)g_2(x),u^2l_q(x)r(x)\right)\right\},
\end{aligned} \quad (68)$$

where $\deg f(x)=t_1$, , $\deg g_1(x)=t_2$, $\deg p(x)=t_3$, $a_1'(x)=\frac{g_1(x)}{a_1(x)}$, $q'(x)=\frac{p(x)}{q(x)}$, $r'(x)=\frac{q(x)}{r(x)}$, $l_f(x)f(x)=x^{\alpha}-1$, $l_{g1}(x)g_1(x)=x^{\beta}-1$, $l_p(x)p(x)=l_q(x)q(x)=x^{\theta}-1$. Then

$$D=D_1\cup D_2\cup D_3\cup D_4\cup D_5\cup D_6 \quad (69)$$

is a minimal spanning set for C, and

$$\left|C\right|=2^{\alpha-t_1}\cdot 4^{\beta-t_2}\cdot 8^{\theta-t_3}\cdot 2^{\deg a_1'(x)}\cdot 4^{\deg q'(x)}.2^{\deg r'(x)} \quad (70)$$

Proof Let $c(x)$ be a any codeword in C. There exist some polynomials $e_1(x),e_2(x),e_3(x)\in S[x]$, such that $c(x)$ can be represented by

$$\begin{aligned}c(x)&=e_1(x)*\left(f(x),0,0\right)+e_2(x)*\left(f_1(x),g_1(x)+ua_1(x),0\right)\\ &+e_3(x)*\left(f_2(x),g_2(x),p(x)+uq(x)+u^2r(x)\right)\end{aligned} \quad (71)$$

If

$\deg e_1(x)<\deg l_f(x)$, then

$$e_1(x)*\left(f(x),0,0\right)\in\langle D_1\rangle \quad (72)$$

Otherwise, have

$$e_1(x)=l_f(x)q_1(x)+r_1(x) \quad (73)$$

where $r_1(x)=0$ or $\deg r_1(x)<\deg l_f(x)$. Hence,

$$\begin{aligned}&e_1(x)*\left(f(x),0,0\right)=l_f(x)q_1(x)+r_1(x)*\left(f(x),0,0\right)\\ &=r_1(x)*\left(f(x),0,0\right)\in\langle D_1\rangle.\end{aligned} \quad (74)$$

If $\deg e_2(x)<\deg l_f(x)$, then

$$e_2(x)*\left(f_1(x),g_1(x)+ua_1(x),0\right)\in\langle D_2\rangle \quad (75)$$

Otherwise, have

$$e_2(x)=l_{g1}(x)q_2(x)+r_2(x) \quad (76)$$

where $r_2(x)=0$ or $\deg r_2(x)=\deg l_{g_1}(x)$. Hence,

$$\begin{aligned}&e_2(x)*\left(f_1(x),g_1(x)+ua_1(x),0\right)\\ &=l_{g1}(x)q_2(x)+r_2(x)*\left(f_1(x),g_1(x)+ua_1(x),0\right)\\ &=l_{g1}(x)q_2(x)*\left(f_1(x),g_1(x)+ua_1(x),0\right)\\ &+r_2(x)*\left(f_1(x),g_1(x)+ua_1(x),0\right)\\ &=q_2(x)*\left(l_{g1}(x)f_1(x),l_{g1}(x)ua_1(x),0\right)\\ &+r_2(x)*\left(f_1(x),g_1(x)+ua_1(x),0\right)\end{aligned} \quad (77)$$

where $r_2(x)*\left(f_1(x),g_1(x)+ua_1(x),0\right)\in\langle D_2\rangle$.

For

$$q_2(x)*\left(l_{g1}(x)f_1(x),l_{g1}(x)ua_1(x),0\right) \quad (78)$$

if $\deg q_2(x)<\deg a'_1(x)$, then

$$q_2(x)*\left(l_{g1}(x)f_1(x),ul_{g1}(x)a_1(x),0\right)\in\langle D_4\rangle \quad (79)$$

Otherwise, have

$$q_2(x) = q_3(x)a_1^{'}(x) + r_3(x) \tag{80}$$

where $r_3(x) = 0$ or $\deg r_3(x) < \deg a'_1(x)$. Hence,

$$\begin{aligned} &q_2(x)*(l_{g1}(x)f_1(x), ul_{g1}(x)a_1(x), 0) \\ &= (q_3(x)a_1^{'}(x) + r_3(x))*(l_{g1}(x)f_1(x), ul_{g1}(x)a_1(x), 0) \\ &= q_3(x)*(l_{g1}(x)f_1(x)a_1^{'}(x), 0, 0) \\ &+ r_3(x)*(l_{g1}(x)f_1(x), ul_{g1}(x)a_1(x), 0) \end{aligned} \tag{81}$$

Here,

$$r_3(x)*(l_{g1}(x)f_1(x), ul_{g1}(x)a_1(x), 0) \in \langle D_4 \rangle \tag{82}$$

and since

$$f(x) \left| \frac{x^\beta - 1}{a_1(x)} f_1(x) \right. \tag{83}$$

$$q_3(x)*(l_{g1}(x)f_1(x)a_1^{'}(x), 0, 0) = q_3(x)*\left(\frac{x^\beta - 1}{a_1(x)} f_1(x), 0, 0\right) \in \langle D_1 \rangle \tag{84}$$

Finally, for $e_3(x)*(f_2(x), g_2(x), p(x) + uq(x) + u^2 r(x))$, if $\deg e_3(x) < \deg l_p(x)$, then

$$e_3(x)*(f_2(x), g_2(x), p(x) + uq(x) + u^2 r(x)) \in \langle D_1 \rangle \tag{85}$$

$$e_3(x) = l_p(x)q_4(x) + r_4(x) \tag{86}$$

where $r_4(x) = 0$ or $\deg r_4(x) < \deg l_p(x)$.

Hence,

Otherwise,

$$\begin{aligned} &e_3(x)*(f_2(x), g_2(x), p(x) + uq(x) + u^2 r(x)) \\ &= l_p(x)q_4(x) + r_4(x)*(f_2(x), g_2(x), p(x) + uq(x) + u^2 r(x)) \\ &= q_4(x)*(l_p(x)f_2(x), l_p(x)g_2(x), ul_p(x)q(x) + u^2 l_p(x)r(x)) \\ &+ r_4(x)*(f_2(x), g_2(x), p(x) + uq(x) + u^2 r(x)). \end{aligned} \tag{87}$$

Here,

$$r_4(x)*(f_2(x), g_2(x), p(x) + uq(x) + u^2 r(x)) \in \langle D_3 \rangle \tag{88}$$

For

$$q_4(x)*(l_p(x)f_2(x), l_p(x)g_2(x), ul_p(x)q(x) + u^2 l_p(x)r(x)) \tag{89}$$

if $\deg q_4(x) < \deg q'(x)$, then

$$q_4(x)*(l_p(x)f_2(x), l_p(x)g_2(x), ul_p(x)q(x) + u^2 l_p(x)r(x)) \in \langle D_5 \rangle \tag{90}$$

Otherwise, have

$$q_4(x) = q^{'}(x)q_5(x) + r_5(x) \tag{91}$$

where $r_5(x) = 0$ or $\deg r_5(x) < \deg q^{'}(x)$.

Hence,

$$\begin{aligned}&q_4(x)*\left(l_p(x)f_2(x),l_p(x)g_2(x),ul_p(x)q(x)+u^2l_p(x)r(x)\right)\\&=q'(x)q_5(x)\\&+r_5(x)*\left(l_p(x)f_2(x),l_p(x)g_2(x),ul_p(x)q(x)+u^2l_p(x)r(x)\right)\\&=q_5(x)*\left(l_p(x)f_2(x),l_p(x)g_2(x),u^2l_p(x)r(x)\right)\\&+r_5(x)*\left(l_p(x)f_2(x),l_p(x)g_2(x),ul_p(x)q(x)+u^2l_p(x)r(x)\right)\end{aligned} \tag{92}$$

Here, $r_5(x)*\left(l_p(x)f_2(x),l_p(x)g_2(x),ul_p(x)q(x)+u^2l_p(x)r(x)\right)\in\langle D_5\rangle$.

For $q_5(x)*\left(l_p(x)f_2(x),l_p(x)g_2(x),u^2l_p(x)r(x)\right)$, if $\deg\left(q_5(x)\right)\langle\deg\left(r'(x)\right)$, then

$$q_5(x)*\left(l_p(x)f_2(x),l_p(x)g_2(x),u^2l_p(x)r(x)\right)\in\langle D_6\rangle \tag{93}$$

Otherwise

$$q_5(x)=r'(x)q_6(x)+r_6(x) \tag{94}$$

where $r_6(x)=0$ or $\deg r_6(x)<\deg r'(x)$.

Hence,

$$\begin{aligned}&r'(x)q_6(x)+r_6(x)*(l_p(x)f_2(x),l_p(x)g_2(x),u^2l_p(x)r(x))\\&=q_6(x)*(\frac{x^\theta-1}{r(x)}f_2(x),\frac{x^\theta-1}{r(x)}g_2(x),0)\\&+r_6(x)*(l_p(x)f_2(x),l_p(x)g_2(x),u^2l_p(x)r(x))\end{aligned} \tag{95}$$

Have

$$r_6(x)*(l_p(x)f_2(x),l_p(x)g_2(x),u^2l_p(x)r(x))\in\langle D_6\rangle \text{ (96).}$$

In this following, consider the

$$q_6(x)*(\frac{x^\theta-1}{r(x)}f_2(x),\frac{x^\theta-1}{r(x)}g_2(x),0) \tag{97}$$

From lemma 5, there exists $k(x)\in S[x]$ such that

$$k(x)(g_1(x)+ua_1(x))=\frac{x^\theta-1}{r(x)}g_2(x)\bmod u^2 \tag{98}$$

and

$$f(x)\,|\,(k(x)f_1(x)+\frac{x^\theta-1}{r(x)}f_2(x))\bmod u \tag{99}$$

Therefore, there is $\lambda(x)$ such that

$$f(x)\lambda(x)=k(x)f_1(x)+\frac{x^\theta-1}{r(x)}f_2(x) \tag{100}$$

which implies that,

$$\frac{x^\theta-1}{r(x)}f_2(x)=f(x)\lambda(x)-k(x)f_1(x) \tag{101}$$

Hence,

$$\begin{aligned}&q_6(x)*(\frac{x^\theta-1}{r(x)}f_2(x),\frac{x^\theta-1}{r(x)}g_2(x),0)\\&=q_6(x)*(f(x)\lambda(x)-k(x)f_1(x),k(x)(g_1(x)+ua_1(x),0))\\&=q_6(x)\lambda(x)(f(x),0,0)+q_6(x)k(x)(f_1(x),g_1(x)+ua_1(x),0)\\&\in\langle D_1\cup D_2\rangle.\end{aligned} \tag{102}$$

Example 1 Let C be a Z_2RS -additive cyclic code in generated $Z_2[x]/(x-1)\times R[x]/(x-1)\times S[x]/(x^5-1)$ by

$$\langle\ (f(x),0,0),(f_1(x),g_1(x)+ua_1(x),0),\ (f_2(x),g_2(x),p(x)+uq(x)+u^2r(x))\ \rangle,$$

where $f(x)=1+x$, $f_1(x)=f_2(x)=1$, $f_1(x)=f_2(x)=1$, $g_2(x)=1$, $a_1(x)=1$, $p(x)=1+x$, $q(x)=1$, $r(x)=1$. In this following,

$$\begin{aligned}&f(x)l_f(x)=x-1\Rightarrow l_f(x)=1,\\&p(x)l_p(x)=x^5-1\Rightarrow l_p(x)=x^4+x^3+x^2+1,\\&q(x)l_q(x)=x^5-1\Rightarrow l_q(x)=x^5-1,\\&g_1(x)l_{g_1}(x)=x-1\Rightarrow l_{g_1}(x)=1.\end{aligned}$$

Hence, from Theorem 3, the generator matrix for C is as follows

$$\begin{pmatrix} 1 & 1 & 1+u+u^2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1+u+u^2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1+u+u^2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1+u+u^2 & 1 \\ 1 & u & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & u+u^2 & 0 & u+u^2 & u+u^2 & u+u^2 \end{pmatrix}$$

Furthermore, the $\Phi(C)$ is a [18, 6, 3] linear binary code.

There is a table about some examples. The binary codes with these parameter are not nice codes. But the binary codes which are marked with * are only about 1 less in minimum distance than the corresponding nice codes.

Table 1. *The codes of binary images of C.*

Generator	α,β,θ	Binary Images
$f(x)=1+x$, $f_1(x)=f_2(x)=1$, $g_1(x)=1+x, g_2(x)=1$, $a_1(x)=1, p(x)=1+x$, $q(x)=1+x, r(x)=1$.	1, 1, 1	[6, 2, 3]*
$f(x)=1+x$, $f_1(x)=f_2(x)=1$, $g_1(x)=1+x^3$, $g_2(x)=1+x+x^3$, $a_1(x)=1+x, p(x)=1+x,$ $q(x)=1, r(x)=1$.	1, 3, 1	[10, 3, 4]*
$f(x)=1+x$, $f_1(x)=f_2(x)=1$, $g_1(x)=1+x^3$, $g_2(x)=1+x+x^3$, $a_1(x)=1+x, p(x)=1$, $q(x)=1, r(x)=1$.	1, 3, 1	[10, 4, 3]*
$f(x)=1+x, f_1(x)=f_2(x)=1$, $g_1(x)=1+x, g_2(x)=1$, $a_1(x)=1, p(x)=1+x$, $q(x)=1, r(x)=1$.	5, 1, 1	[10, 6, 2]*
$f(x)=1+x$, $f_1(x)=f_2(x)=1$, $g_1(x)=1+x^5$, $g_2(x)=1+x+x^2+x^3+x^4$, $a_1(x)=1+x+x^2+x^3+x^4$, $p(x)=1+x, q(x)=1, r(x)=1$.	1, 5, 1	[14, 2, 8]*
$f(x)=1+x$, $f_1(x)=f_2(x)=1$, $g_1(x)=1+x^3, g_2(x)=1+x,$ $a_1(x)=1+x+x^2$, $p(x)=1+x^3$, $q(x)=1+x, r(x)=1$.	1, 3, 3	[16, 4, 6]
$f(x)=1+x$, $f_1(x)=f_2(x)=1$, $g_1(x)=1+x+x^2+x^3+x^4$, $g_2(x)=1+x+x^2+x^3$, $a_1(x)=1+x+x^2+x^3$, $p(x)=1+x, q(x)=1+x, r(x)=1$.	1, 7, 1	[18, 5, 6]
$f(x)=1+x, f_1(x)=1$, $f_2(x)=0, g_1(x)=1+x^7$, $g_2(x)=1+x+x^2+x^3+x^4+x^5+x^6$, $a_1(x)=1+x^7, p(x)=1+x,$ $q(x)=1, r(x)=1$.	1, 7, 1	[18, 2, 10]
$f(x)=1+x$, $f_1(x)=f_2(x)=1$, $g_1(x)=1+x+x^2+x^4$, $g_2(x)=1+x^2+x^3$, $a_1(x)=1+x, p(x)=1+x,$ $q(x)=1+x, r(x)=1$.	1, 7, 1	[18, 7, 4]
$f(x)=1+x+x^2+x^4$, $f_1(x)=f_2(x)=1+x^2+x^3,$ $g_1(x)=1+x^3, g_2(x)=1+x,$ $a_1(x)=1+x+x^2, p(x)=1+x^3,$ $q(x)=1+x, r(x)=1$.	7, 3, 3	[22, 7, 4]

4. Conclusion

In this paper, we have studied additive cyclic codes over the ring Z_2RS, where $R=Z_2+uZ_2$, $u^2=0$ mod 2, and $S=Z_2+uZ_2+u^2Z_2$, $u^2=0$ mod 2. We have given the definition of $Z_2(Z_2+uZ_2)(Z_2+uZ_2+u^2Z_2)$ -additive codes with generator matrices and parity-check matrices. Furthermore, the fundamental results on the generator polynomials and spanning sets for these additive cyclic codes of block length (α,β,θ), where θ is odd have been obtained. Finding Z_2RS -additive cyclic code and its dual of arbitrary block length （α,β,θ） may be an interesting problem.

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